Problem 1.3

*Cosine and sine by vector algebra**
Find the cosine and the sine of the angle between $\mathbf{A} = (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\mathbf{B} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$.

Solution

The dot product is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta,$$

where $|\mathbf{A}|$ and $|\mathbf{B}|$ are the magnitudes of $\mathbf{A}$ and $\mathbf{B}$, respectively, and θ is the angle between the vectors. This angle must be between 0° and 180°. Solve for $\cos\theta$.

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

Now plug in the numbers.

$$\cos\theta = \frac{(1)(-2) + (1)(1) + (1)(1)}{\sqrt{3^2 + 1^2 + 1^2}\sqrt{(-2)^2 + 1^2 + 1^2}}$$

Therefore,

$$\cos\theta = -\frac{4}{\sqrt{66}} \approx -0.492.$$

Sine and cosine are related by the formula,

$$\sin^2\theta + \cos^2\theta = 1.$$

Solve this for $\sin\theta$.

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

Since θ is between 0° and 180°, the sine of the angle is positive, so

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(-\frac{4}{\sqrt{66}}\right)^2}.$$

Therefore,

$$\sin\theta = \frac{5}{\sqrt{33}} \approx 0.870.$$